2 Inverse Trigonometric Functions

Teaching-Learning Points

- The sine function is defined as

  $\sin : R \to [-1, 1]$

  Which is not a one-one function over the whole domain and hence its inverse does not exist but if we restrict the domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ then the sine function becomes a one-one and onto function and therefore we com define the inverse of the function $\sin : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ as $\sin^{-1} : [-1,1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

  In fact there are other intervals also like $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc which may also be taken as range of the function $\sin^{-1}$. Corresponding to each interval we get branch of $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called principal value branch similiary for other inverse trigonometric functions we have principal value branches.

- List of principal value branches and the domain of inverse trigonometric functions.

Functions	**Domain**	**Range (Principal value Branch)**
$y = \sin^{-1}x$	$-1 \le x \le 1$	$\frac{-\pi}{2} \le y \le \pi/2$
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1}x$	$-\infty < x < \infty$	$\frac{-\pi}{2} < y < \frac{\pi}{2}$
$y = \cot^{-1}x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \sec^{-1}x$	$\begin{cases} -\infty < x \le -1 \\ 1 \le x < \infty \end{cases}$	$\frac{\pi}{2} < y \le \pi$ $0 \le y < \frac{\pi}{2}$
$y = \operatorname{cosec}^{-1}x$	$\begin{cases} -\infty < x \le -1 \\ 1 \le x < \infty \end{cases}$	$\frac{-\pi}{2} \le y < 0$ $0 < y \le \frac{\pi}{2}$

l **Properties of inverse trigonometric functions :**

1. (*i*) $\sin^{-1}(\sin x) = x, \quad x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

   (*ii*) $\sin(\sin^{-1} x) = x, \quad x \in [-1,1]$

   (*iii*) $\cos^{-1}(\cos x) = x, \quad x \in [0, \pi]$

   (*iv*) $\cos(\cos^{-1} x) = x, \quad x \in [-1,1]$

   (*v*) $\tan^{-1}(\tan x) = x, \quad x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

   (*vi*) $\tan(\tan^{-1} x) = x, \quad x \in \mathrm{R}.$

2. (*i*) $\sin^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1} x, \quad |x| \geq 1$

   (*ii*) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, \quad |x| \geq 1$

   (*iii*) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x, \quad x > 0$

3. (*i*) $\sin^{-1}(-x) = -\sin^{-1} x, \quad x \in [-1, 1]$

   (*ii*) $\tan^{-1}(-x) = -\tan^{-1} x, \quad x \in \mathrm{R}$

   (*iii*) $\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1} x, \quad |x| \geq 1$

   (*iv*) $\cos^{-1}(-x) = \pi - \cos^{-1} x, \quad x \in [-1, 1]$

   (*v*) $\sec^{-1}(-x) = \pi - \sec^{-1} x, \quad |x| \geq 1$

   (*vi*) $\cot^{-1}(-x) = \pi - \cot^{-1} x, \quad x \in \mathrm{R}$

4. (*i*) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1,1]$

   (*ii*) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in \mathrm{R}.$

   (*iii*) $\mathrm{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}, |x| \geq 1$

5. (*i*) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, xy < 1$

   (*ii*) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, xy > -1$

6. (*i*) $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}, |x| < 1$

(*ii*) $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2}, |x| \le 1$

(*iii*) $2\tan^{-1}x = \cos^{-1}\dfrac{1-x^2}{1+x^2}, x \ge 0$

Question for Practice

Evaluate the following Integrals

Very Short Answer Type Questions (1 Mark)

Q1. Write the principal value of $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$.

Q2. Write the principal value of $\text{cosec}^{-1}(-\sqrt{2})$.

Q3. Write the principal value of $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$.

Q4. Write the principal value of $\tan^{-1}(-\sqrt{3})$.

Q5. Write the principal value of $\sec^{-1}(-\sqrt{2})$.

Q6. Write the principal value of $\cos^{-1}\left(\dfrac{1}{2}\right)$.

Q7. Show that $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$.

Q8. Show that $\cos^{-1}x = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$.

Q9. Show that $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$.

Q10. Show that $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

Q11. Show that $\cos^{-1} x = 2\sin^{-1}\sqrt{\frac{1-x}{2}}$.

Q12. Write $\sin^{-1}(3x - 4x^3)$ in the simplest form.

Q13. Write $\cos^{-1}(4x^3 - 3x)$ in the simplest form.

Q14. Evaluate $\text{cosec}^{-1}\left\{\text{cosec}\left(\frac{-\pi}{4}\right)\right\}$.

Q15. Evaluate $\cos\left\{\frac{\pi}{3} - \cos^{-1}\left(\frac{1}{2}\right)\right\}$.

Q16. Show that $\cos^{-1} x = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

Q17. Write $\cos^{-1}(2x^2 - 1)$ in the simplest form.

Q18. Write $\cos^{-1}(1 - 2x^2)$ in the simplest form.

Q19. Write $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$, $0 \le x < \pi$.

Q20. Show that $\sin^{-1} 2x\sqrt{1-x^2} = 2\sin^{-1} x$.

Q21. Evaluate : $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right\}$.

Q22. Evaluate : $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

Q23. Find x, if $\tan^{-1} x = \pi/4$.

Q24. Evaluate $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Q25. Evaluate $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

Q26. Evaluate $\sin^{-1}(\sin 2\pi/3)$.

Q27. Evaluate $\text{cosec}^{-1}\left\{\text{cosec}\frac{3\pi}{4}\right\}$.

Q28. Evaluate $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)$.

Q29. Write $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, |x| < a$ in the simplest form.

Q30. Find x, if $\cot^{-1} x + \tan^{-1} 7 = \frac{\pi}{2}$.

Q31. Find x, if $\sin^{-1} x = \frac{\pi}{6} + \cos^{-1} x$.

Q32. Find x, if $4\sin^{-1} x = \pi - \cos^{-1} x$.

Q33. Find x, if $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$.

Q34. Write $\sin^{-1}\left(\frac{2x}{1+x^2}\right) -1 \le x \le 1$, in the simplest form.

Q35. Write $\sin^{-2}\left(2x\sqrt{1-x^2}\right)$ in the simplest form.

Short Answer Questions Carrying 4 Marks each

Q36. Solve for x : $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$.

Q37. Solve for x : $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$.

Q38. If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, prove that $a + b + c = abc$.

Q39. Solve for x : $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x, x > 0$.

Q40. Solve for x : $\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = -\tan^{-1} 7$.

Q41. Solve for x : $\tan^{-1}\left(\frac{2x}{1+x^2}\right) + \cot^{-1}\left(\frac{1+x^2}{2x}\right) = \frac{-\pi}{2}$.

Q42. Solve for x : $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\frac{23}{36}$.

Q43. Solve for x : $\sin^{-1}\frac{8}{17} = \sin^{-1} x - \sin^{-1}\frac{3}{5}$.

Q44. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = n\pi + \frac{3\pi}{4}$.

Q45. Solve for x : $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) - \tan^{-1} 3x = 0$.

Q46. Prove that $\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi.$

Q47. Prove that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}.$

Q48. Prove that $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}.$

Q49. Prove that $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}.$

Q50. Prove that $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\left(\frac{253}{325}\right).$

Q51. Prove that $\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{2}.$

Q52. Prove that $\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{27}{11}\right).$

Q53. Prove that $\cos^{-1}\left(\frac{63}{65}\right)+2\tan^{-1}\left(\frac{1}{5}\right)=\sin^{-1}\left(\frac{3}{5}\right).$

Q54. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right).$

Q55. Prove that : $2\tan^{-1}\left(\frac{5}{12}\right)-\tan^{-1}\left(\frac{1}{70}\right)+\tan^{-1}\left(\frac{1}{99}\right)=\frac{\pi}{4}.$

Q56. Prove that : $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}.$

Q57. Prove that $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right).$

Q58. Prove that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right).$

Q59. Prove that : $2\tan^{-1}\frac{1}{5}+\operatorname{cosec}^{-1}5\sqrt{2}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}.$

Q60. Prove that : $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}.$

Answers

1. $\frac{\pi}{3}$
2. $\frac{-\pi}{4}$
3. $\frac{2\pi}{3}$
4. $\frac{-\pi}{3}$
5. $\frac{3\pi}{4}$
6. $\frac{\pi}{3}$
12. $3\sin^{-1}x$
13. $3\cos^{-1}x$
14. $\frac{-\pi}{4}$
15. 1
17. $2\cos^{-1}x$
18. $2\sin^{-1}x$
19. $\frac{x}{2}$
21. 1
22. π
23. $\frac{\pi}{4}$
24. $\frac{-\pi}{4}$
25. $\frac{5\pi}{6}$
26. $\pi/3$
27. $\pi/4$
28. $\pi/3$
29. $\sin^{-1}\left(\frac{x}{a}\right)$
30. 7
31. $\frac{\sqrt{3}}{2}$
32. $\frac{1}{2}$
33. $\sqrt{3}$
34. $2\tan^{-1}x$
35. $2\sin^{-1}x$
36. $-8, \frac{1}{4}$
37. $\frac{1}{6}$
39. $\frac{1}{\sqrt{3}}$
40. $x = 2$
41. $x = 1$
42. $\frac{-3}{8}, \frac{4}{3}$
43. $\frac{77}{85}$
44. $\frac{-1}{6}, 1$
45. $0, \frac{1}{2}, -\frac{1}{2}$