Exercise 5.4

Question 1:

Differentiate the following w.r.t. x: $\frac{e^x}{\sin x}$

Answer

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$

$$= \frac{\sin x.\left(e^x\right) - e^x \cdot \left(\cos x\right)}{\sin^2 x}$$

$$= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$$

Question 2:

Differentiate the following w.r.t. x: $e^{\sin^{-1} x}$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx}\left(\sin^{-1} x\right)$$

$$= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in \left(-1, 1\right)$$

Question 3:

Differentiate the following w.r.t. x:

e^{x^3}

Answer

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right) = e^{x^3} \cdot \frac{d}{dx}\left(x^3\right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

Question 4:

Differentiate the following w.r.t. x:

$\sin\left(\tan^{-1} e^{-x}\right)$

Answer

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{1}{1+\left(e^{-x}\right)^2} \cdot \frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times (-1)$$

$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. x:

$\log\left(\cos e^x\right)$

Answer

Let $y = \log\left(\cos e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$

$$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}\left(\cos e^x\right)$$

$$= \frac{1}{\cos e^x} \cdot \left(-\sin e^x\right) \cdot \frac{d}{dx}\left(e^x\right)$$

$$= \frac{-\sin e^x}{\cos e^x} \cdot e^x$$

$$= -e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{N}$$

Question 6:

Differentiate the following w.r.t. x:

$e^x + e^{x^2} + \ldots + e^{x^5}$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$

$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$

$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$

$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$

$$= e^x + 2xe^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$$

Question 7:

Differentiate the following w.r.t. x:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer

Let

$y=\sqrt{e^{\sqrt{x}}}$

$y^2=e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$$y^2=e^{\sqrt{x}}$$

$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$

$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},\ x>0$$

Question 8:

Differentiate the following w.r.t. x:

$\log(\log x), x>1$

Answer

Let $y=\log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]$$
$$=\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$
$$=\frac{1}{\log x}\cdot\frac{1}{x}$$
$$=\frac{1}{x\log x}$$, x > 1

Question 9:

Differentiate the following w.r.t. x:

$\frac{\cos x}{\log x}, x>0$

Answer

Let $y=\frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\times\log x-\cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$
$$=\frac{-\sin x\log x-\cos x\times\frac{1}{x}}{(\log x)^2}$$
$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2}, x>0$$

Question 10:

Differentiate the following w.r.t. x:

$\cos(\log x+e^x), x>0$

Answer

Let $y=\cos(\log x+e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin\left(\log x + e^x\right) \cdot \frac{d}{dx}\left(\log x + e^x\right)$$

$$= -\sin\left(\log x + e^x\right) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}\left(e^x\right)\right]$$

$$= -\sin\left(\log x + e^x\right) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin\left(\log x + e^x\right), x > 0$$